Exercise 2.5.29

Solve Laplace's equation inside a circle of radius a,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

subject to the boundary condition

$$u(a,\theta) = f(\theta).$$

Solution

Because the Laplace equation is linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ and plug it into the PDE.

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0 \\
\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}R(r)\Theta(\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}R(r)\Theta(\theta) &= 0 \\
\frac{\Theta(\theta)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Theta}{d\theta^2} &= 0
\end{aligned}$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$ in order to separate variables.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned}
\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= \lambda \\
-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} &= \lambda
\end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter whether the minus sign is grouped with r or θ as long as all eigenvalues are taken into account.

Though it would seem the method of separation of variables would fail because the provided boundary condition is inhomogeneous, there are two others that aren't listed which are homogeneous.

$$\begin{aligned} u(r,0) &= u(r,2\pi) \\ \frac{\partial u}{\partial \theta}(r,0) &= \frac{\partial u}{\partial \theta}(r,2\pi) \end{aligned}$$

These periodic boundary conditions come from the fact that the domain is a disk, and the solution and its slope in the θ-direction must repeat every 2π radians. Generally, the interval of θ used is the one that $f(\theta)$ is defined over. Mr. Haberman uses $-\pi < \theta < \pi$ in the textbook, so that's what will be used here. The following boundary conditions will be used instead.

$$\begin{aligned} u(r,-\pi) &= u(r,\pi) \\ \frac{\partial u}{\partial \theta}(r,-\pi) &= \frac{\partial u}{\partial \theta}(r,\pi) \end{aligned}$$

Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into them.

$$\begin{aligned} u(r,-\pi) = u(r,\pi) \quad &\rightarrow \quad R(r)\Theta(-\pi) = R(r)\Theta(\pi) \quad &\rightarrow \quad \Theta(-\pi) = \Theta(\pi) \\ \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \quad &\rightarrow \quad R(r)\Theta'(-\pi) = R(r)\Theta'(\pi) \quad &\rightarrow \quad \Theta'(-\pi) = \Theta'(\pi) \end{aligned}$$

Now solve the ODE for Θ.

$$\Theta'' = -\lambda\Theta$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$\Theta'' = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos \mu\theta + C_2 \sin \mu\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin \mu\theta + C_2 \cos \mu\theta)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta(-\pi) = C_1 \cos \mu\pi - C_2 \sin \mu\pi &= C_1 \cos \mu\pi + C_2 \sin \mu\pi = \Theta(\pi) \\ \Theta'(-\pi) = \mu(C_1 \sin \mu\pi + C_2 \cos \mu\pi) &= \mu(-C_1 \sin \mu\pi + C_2 \cos \mu\pi) = \Theta'(\pi) \end{aligned}$$

Cancel the terms common to both sides.

$$\begin{aligned} -C_2 \sin \mu\pi &= C_2 \sin \mu\pi \\ \mu(C_1 \sin \mu\pi) &= \mu(-C_1 \sin \mu\pi) \end{aligned}$$

To avoid the trivial solution, we insist that $C_1 \neq 0$ and $C_2 \neq 0$.

$$\begin{aligned} \sin \mu\pi &= 0 \\ \mu\pi &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= n \end{aligned}$$

There are positive eigenvalues $\lambda = n^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos \mu\theta + C_2 \sin \mu\theta \quad \rightarrow \quad \Theta_n(\theta) = E \cos n\theta + F \sin n\theta.$$

Using $\lambda = n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2 R'' + rR' - n^2 R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2 m(m-1)r^{m-2} + rmr^{m-1} - 9n^2 r^m = 0$$

$$m(m-1)r^m + mr^m - n^2 r^m = 0$$

$$m(m-1) + m - n^2 = 0$$

$$m^2 - n^2 = 0$$

$$(m+n)(m-n) = 0$$

$$m = \{-n, n\}$$

Two solutions to the ODE are $R = r^{-n}$ and $R = r^n$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-n} + Br^n$$

In order for the solution to remain finite as $r \rightarrow 0$, set $A = 0$.

$$R(r) = Br^n$$

Now check to see if zero is an eigenvalue: $\lambda = 0$.

$$\Theta'' = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the boundary conditions here to determine C_3 and C_4.

$$\begin{aligned} \Theta(-\pi) = -C_3\pi + C_4 &= C_3\pi + C_4 = \Theta(\pi) \\ \Theta'(-\pi) = C_3 &= C_3 = \Theta'(\pi) \end{aligned}$$

C_4 cancels from the first equation, leaving $-C_3\pi = C_3\pi$. Only $C_3 = 0$ satisfies this equation, and C_4 remains arbitrary.

$$\Theta(\theta) = C_4$$

This is not the trivial solution, so zero is an eigenvalue. The eigenfunction associated with it is a constant. Now solve the ODE for R with $\lambda = 0$.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Multiply both sides by R/r.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = D_1$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{D_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = D_1 \ln r + D_2$$

In order for the solution to remain finite as $r \to 0$, set $D_1 = 0$.

$$R(r) = D_2$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\Theta'' = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh\gamma\theta + C_6 \sinh\gamma\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \gamma(C_5 \sinh\gamma\theta + C_6 \cosh\gamma\theta)$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\begin{gathered}\Theta(-\pi) = C_5 \cosh\gamma\pi - C_6 \sinh\gamma\pi = C_5 \cosh\gamma\pi + C_6 \sinh\gamma\pi = \Theta(\pi) \\ \Theta'(-\pi) = \gamma(-C_5 \sinh\gamma\pi + C_6 \cosh\gamma\pi) = \gamma(C_5 \sinh\gamma\pi + C_6 \cosh\gamma\pi) = \Theta'(\pi)\end{gathered}$$

Cancel the terms common to both sides in each equation.

$$\begin{gathered}-C_6 \sinh\gamma\pi = C_6 \sinh\gamma\pi \\ \gamma(-C_5 \sinh\gamma\pi) = \gamma(C_5 \sinh\gamma\pi)\end{gathered}$$

There are no nonzero values of γ that can satisfy these equations, which means the only way they are is if $C_5 = 0$ and $C_6 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions $u = R_n(r)\Theta_n(\theta)$ over all the eigenvalues.

$$u(r,\theta) = B_0 + \sum_{n=1}^{\infty} B_n r^n (E_n \cos n\theta + F_n \sin n\theta)$$

The aim now is to use the prescribed boundary condition to determine all of the coefficients.

$$u(a,\theta) = B_0 + \sum_{n=1}^{\infty} B_n a^n (E_n \cos n\theta + F_n \sin n\theta) = f(\theta) \tag{1}$$

To get B_0, integrate both sides with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi} \left[B_0 + \sum_{n=1}^{\infty} B_n a^n (E_n \cos n\theta + F_n \sin n\theta) \right] d\theta = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

Split up the integral on the left and bring the constants in front.

$$B_0 \underbrace{\int_{-\pi}^{\pi} d\theta}_{=2\pi} + \sum_{n=1}^{\infty} B_n a^n \left(E_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}_{=0} + F_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}_{=0} \right) = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

Evaluate the integrals.

$$B_0(2\pi) = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

Therefore,

$$\boxed{B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta}$$

To get the next coefficient, multiply both sides of equation (1) by $\cos p\theta$, where p is an integer

$$B_0 \cos p\theta + \sum_{n=1}^{\infty} B_n a^n (E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) = f(\theta) \cos p\theta$$

and then integrate both sides with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi} \left[B_0 \cos p\theta + \sum_{n=1}^{\infty} B_n a^n (E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) \right] d\theta = \int_{-\pi}^{\pi} f(\theta) \cos p\theta\, d\theta$$

Split up the integral on the left and bring the constants in front.

$$B_0 \underbrace{\int_{-\pi}^{\pi} \cos p\theta\, d\theta}_{=0} + \sum_{n=1}^{\infty} B_n a^n \left(E_n \int_{-\pi}^{\pi} \cos n\theta \cos p\theta\, d\theta + F_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta \cos p\theta\, d\theta}_{=0} \right) = \int_{-\pi}^{\pi} f(\theta) \cos p\theta\, d\theta$$

Because the sine and cosine functions are orthogonal, the third integral on the left is zero. The cosine functions are orthogonal with one another, so the second integral is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n a^n \left(E_n \int_{-\pi}^{\pi} \cos^2 n\theta\, d\theta \right) = \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta$$

Evaluate the integral.

$$a^n B_n E_n(\pi) = \int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta$$

Therefore,

$$\boxed{B_n E_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta.}$$

To get the final coefficient, multiply both sides of equation (1) by $\sin p\theta$, where p is an integer

$$B_0 \sin p\theta + \sum_{n=1}^{\infty} B_n a^n (E_n \cos n\theta \sin p\theta + F_n \sin n\theta \sin p\theta) = f(\theta)\sin p\theta$$

and then integrate both sides with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[B_0 \sin p\theta + \sum_{n=1}^{\infty} B_n a^n (E_n \cos n\theta \sin p\theta + F_n \sin n\theta \sin p\theta)\right] d\theta = \int_{-\pi}^{\pi} f(\theta)\sin p\theta\, d\theta$$

Split up the integral on the left and bring the constants in front.

$$B_0 \underbrace{\int_{-\pi}^{\pi} \sin p\theta\, d\theta}_{=\,0} + \sum_{n=1}^{\infty} B_n a^n \left(E_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \sin p\theta\, d\theta}_{=\,0} + F_n \int_{-\pi}^{\pi} \sin n\theta \sin p\theta\, d\theta\right) = \int_{-\pi}^{\pi} f(\theta)\sin p\theta\, d\theta$$

Because the sine and cosine functions are orthogonal, the second integral on the left is zero. The sine functions are orthogonal with one another, so the third integral is zero if $n \neq p$. Only if $n = p$ does the integral yield a nonzero result.

$$B_n a^n \left(F_n \int_{-\pi}^{\pi} \sin^2 n\theta\, d\theta\right) = \int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta$$

Evaluate the integral.

$$a^n B_n F_n(\pi) = \int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta$$

Therefore,

$$\boxed{B_n F_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta.}$$

With these boxed formulas for the coefficients, the general solution for the PDE is better written as

$$\boxed{u(r,\theta) = B_0 + \sum_{n=1}^{\infty} r^n (B_n E_n \cos n\theta + B_n F_n \sin n\theta).}$$